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Characterization of the trace by monotonicity inequalities ☆

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Abstract

Let φ be a positive linear functional on the algebra of $n \times n$ complex matrices and p be a number greater than 1. The main result of the paper says that if for any pair A, B of positive semi-definite $n \times n$ matrices with $A \leqslant B$ the inequality $\varphi(A^p) \leqslant \varphi(B^p)$ holds true, then φ is a nonnegative scalar multiple of the trace.

Keywords: Characterization of the trace; Monotonicity

Throughout the paper, $\mathscr{M}_n$ stands for the algebra of $n \times n$ complex matrices, $\mathscr{M}_n^{\mathrm{h}}$ and $\mathscr{M}_n^{+}$ denote the subsets of Hermitian and positive semi-definite matrices, respectively. For $A, B \in \mathscr{M}_n^{\mathrm{h}}$, the notation $A \leqslant B$ means that $B - A \in \mathscr{M}_n^{+}$. A linear functional φ on $\mathscr{M}_n$ is said to be *positive* if $\varphi(A) \geqslant 0$ for all $A \in \mathscr{M}_n^{+}$. For a real-valued function f of a real variable and a matrix $A \in \mathscr{M}_n^{\mathrm{h}}$, the value $f(A)$ is understood by means of the functional calculus for Hermitian matrices.

The Löwner–Heinz inequality says that if $0 \leqslant p \leqslant 1$ then for any pair $A, B \in \mathscr{M}_n^{+}$ such that $A \leqslant B$, it holds $A^p \leqslant B^p$. It is well known that a weaker inequality $\mathrm{Tr}(A^p) \leqslant \mathrm{Tr}(B^p)$ still holds for every $p > 1$. The aim of the present paper is to show that the latter property can serve to characterize the trace among the positive linear functionals on $\mathscr{M}_n$.

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